SOME PROBLEMS OF THE THEORY OF DYNAMIC PROGRAMMING

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The theory of optimum processes in linear systems has been rigorously developed in recent years in the works of Pontriagin, Poltianskii and Gamkrelidze [1, 2, 3].

These studies were preceded by the work of Fel'dbaum [4], Lerner [5, 6] and others. Considerable contribution to this field was made by Bellman [7.8.9], the author of a fundamental paper on dynamic programming [9].

In this paper, one of the problems of the theory of dynamic programming is considered, namely, the problem of choosing the law of variation of the additional external forces by means of which one can ensure realization of prescribed motion in a linear nonstationary system. This problem is considered for both continuous and impulsive systems.

1. Basic continuous systems. The equations of motion of a continuous system can be represented as follows:

$$\sum_{k=1}^{n} f_{jk}(D)\, y_k = x_j(t) + q_j(t) \qquad (j = 1, \ldots, n) \tag{1.1}$$

Here y_k are generalized coordinates, $x_j(t)$ are given external forces, $q_j(t)$ are additional external forces for which the law of variation with respect to time must be chosen such that the prescribed motion will take place; $f_{jk}(D)$ denote polynomials in D the coefficients of which are given functions of time; $D = d/dt$ is the differential operator with respect to time.

It is not difficult to see that the equations of motion (1.1) also apply to systems with the presence of commonly used control forces which are error functions (i.e. the difference between the desired and the actual values of the controlled coordinates of a system) and their derivatives.

The external forces necessary for this are included in the number of the given external forces $x_j(t)$, while the forces which should be functions of controlled coordinates and their derivatives are taken into account in the left-hand side of equations (1.1).

The system of equations (1.1) may be presented as follows:

$$b_{j1}(t)\overset{m_1}{y_1} + b_{j2}(t)\overset{m_2}{y_2} + \ldots + b_{jn}(t)\overset{m_n}{y_n} =$$
$$= \Psi_j\left(\overset{m_1-1}{y_1}, \ldots, y_1, \ldots, \overset{m_n-1}{y_n}, \ldots, y_n\right) + x_j(t) + q_j(t) \qquad (j=1, \ldots, n) \tag{1.2}$$

Here $m_k (k = 1, \ldots, n)$ denote the order of the highest derivative of y_k with respect to time occurring in (1.1). The functions Ψ_j entering (1.2) are linear functions of their arguments.

Assuming that the determinant

$$\Delta^* = |b_{jk}(t)| \tag{1.3}$$

is not identically equal to zero we obtain from equations (1.2)

$$\overset{m_j}{y_j} = \Phi_j\left(\overset{m_1-1}{y_1}, \ldots, y_1, \ldots, \overset{m_n-1}{y_n}, \ldots, y_n\right) +$$
$$+ \frac{B_{1j}(t)}{\Delta^*(t)}[x_1(t) + q_1(t)] + \ldots + \frac{B_{nj}(t)}{\Delta^*(t)}[x_n(t) + q_n(t)] \qquad (j = 1, \ldots, n) \tag{1.4}$$

Here Φ_j are the linear functions of their arguments and B_{ij} are the cofactors of the elements b_{lj} in the determinant (1.3).

The system of equations (1.4) can be reduced to the Cauchy form. To this end let us introduce a new variable z_i by means of the following relationships

$$z_1 = y_1,\ z_2 = \dot{y}_1, \ldots, z_{m_1} = \overset{m_1-1}{y_1}, \ldots, z_r = \overset{m_n-1}{y_n} \tag{1.5}$$

where

$$r = m_1 + m_2 + \ldots + m_n \tag{1.6}$$

Let us denote linear combinations of external forces in the right-hand sides of equations (1.4) as follows:

$$X_{\sigma_j}(t) = \frac{B_{1j}(t)}{\Delta^*(t)}x_1(t) + \ldots + \frac{B_{nj}(t)}{\Delta^*(t)}x_n(t)$$
$$Q_{\sigma_j}(t) = \frac{B_{1j}(t)}{\Delta^*(t)}q_1(t) + \ldots + \frac{B_{nj}(t)}{\Delta^*(t)}q_n(t) \qquad (\sigma_j = \sigma_1, \ldots, \sigma_n) \tag{1.7}$$

where

$$\sigma_1 = m_1, \qquad \sigma_2 = m_1 + m_2, \ldots, \sigma_n = r \tag{1.8}$$

Equations (1.4) can now be written as follows:

$$\dot{z}_1 - z_2 = 0, \ldots, \dot{z}_{m_1} - \Phi_1(z_1, z_2, \ldots, z_r) = X_{\sigma_1}(t) + Q_{\sigma_1}(t), \ldots$$
$$\ldots \dot{z}_r - \Phi_n(z_1, z_2, \ldots, z_r) = X_{\sigma_n}(t) + Q_{\sigma_n}(t) \tag{1.9}$$

By virtue of linearity of the functions $\Phi_j(z_1, z_2, \ldots, z_r)$, equations (1.9) can be represented in the following form

$$\dot{z}_j + \sum_{k=1}^{r} a_{jk}(t) z_k = X_j(t) + Q_j(t) \qquad (j = 1, \ldots, r) \tag{1.10}$$

In equations (1.10) the functions $X_\mu(t)$ and $Q_\mu(t)$ with $\mu \neq \sigma_l$ $(l = 1, \ldots, n)$ are identically equal to zero.

The system of scalar equations (1.10) is equivalent to a matrix equation

$$\dot{z} + a(t) z = X(t) + Q(t) \tag{1.11}$$

where z, $a(t)$, $X(t)$ and $Q(t)$ are the following matrices:

$$z = \|z_j\|, \qquad a(t) = \|a_{jk}(t)\|, \qquad X(t) = \|X_j(t)\|, \qquad Q(t) = \|Q_j(t)\|. \tag{1.12}$$

The general solution of equation (1.11) is of the form:

$$z(t) = N(t, t_0) z(t_0) + \int_{t_0}^{t} N(t, \tau) [X(\tau) + Q(\tau)] d\tau \tag{1.13}$$

where

$$N(t, \tau) = \theta(t) \theta^{-1}(\tau) \tag{1.14}$$

and $\theta(t)$ is the fundamental matrix for the homogeneous matrix equation obtained from (1.11) when $X(t) + Q(t) \equiv 0$. $\theta^{-1}(t)$ denotes the inverse matrix.

The function $N(t, \tau)$ is the matrix weighting function for the system (1.10).

Since the functions $X_\mu(t)$ and $Q_\mu(t)$ are identically equal to zero for $\mu \neq \sigma_l$ $(l = 1, \ldots, n)$, the elements of the z-matrix will be

$$z_j(t) = \sum_{k=1}^{r} N_{jk}(t, t_0) z_k(t_0) + \int_{t_0}^{t} \sum_{i=1}^{n} N_{j\sigma_i}(t, \tau) [X_{\sigma_i}(\tau) + Q_{\sigma_i}(\tau)] d\tau \quad (j = 1, \ldots, r) \tag{1.15}$$

Substituting expressions (1.7) defining $X_{\sigma_i}(t)$ and $Q_{\sigma_i}(t)$ into (1.15) we obtain

$$z_j(t) = \sum_{k=1}^{r} N_{jk}(t, t_0) z_k(t_0) + \tag{1.16}$$

$$+\int_{t_0}^{t}\sum_{l=1}^{n}\sum_{i=1}^{n} N_{j\sigma_i}(t,\tau)\frac{B_{li}(\tau)}{\Delta^*(\tau)}[x_l(\tau)+q_l(\tau)]\,d\tau \quad (j=1,\ldots,r)$$

Denoting by

$$W_{jl}(t,\tau)=\sum_{i=1}^{n} N_{j\sigma_i}(t,\tau)\frac{B_{li}(\tau)}{\Delta^*(\tau)} \qquad \left(\begin{matrix} j=1,\ldots,r \\ l=1,\ldots,n\end{matrix}\right) \tag{1.17}$$

let us represent the general solution of (1.10) in the following form

$$z_j(t)=\sum_{k=1}^{r} N_{jk}(t,t_0)\,z_k(t_0)+\sum_{l=1}^{n}\int_{t_0}^{t} W_{jl}(t,\tau)[x_l(\tau)+q_l(\tau)]\,d\tau \qquad (j=1,\ldots,r) \tag{1.18}$$

Let us now require that some generalized coordinates $z_{p_1}, \ldots, z_{p_m}$, after some instant of time t_1, should vary in accordance with the following laws:

$$z_{p_i}(t)=r_{p_i}(t) \qquad (t\geqslant t_1) \qquad (i=1,\ldots,m) \tag{1.19}$$

It follows from (1.18) that conditions (1.19) will be satisfied if the additional forces $q_l(t)$ are chosen such that for $t \geqslant t_1$ the following relationships hold:

$$\sum_{l=1}^{n}\int_{t_0}^{t} W_{p_il}(t,\tau)\,q_l(\tau)\,d\tau = R_{p_i}(t) \qquad (t\geqslant t_1) \qquad (i=1,\ldots,m) \tag{1.20}$$

where

$$R_{p_i}(t)=r_{p_i}(t)-\sum_{k=1}^{r} N_{p_ik}(t,t_0)\,z_k(t_0)-\sum_{l=1}^{n}\int_{t_0}^{t} W_{p_il}(t,\tau)\,x_l(\tau)\,d\tau \quad (i=1,\ldots,m) \tag{1.21}$$

The relationships (1.20) define the law, in accordance with which the additional forces $g_l(t)$ $(l = 1, \ldots, n)$ must vary. If the number of the generalized coordinates m (the law of variation of which is prescribed) is less than the number of possible additional forces n, we shall take

$$q_{\alpha_1}(t)\equiv q_{\alpha_2}(t)\equiv\ldots\equiv q_{\alpha_{n-m}}(t)\equiv 0 \tag{1.22}$$

From (1.20) we have m relationships for determining m additional external forces $q_{s_1}(t)$, $q_{s_2}(t)$, $\ldots$, $q_{s_m}(t)$:

$$\sum_{\mu=1}^{m}\int_{t_0}^{t} W_{p_is_\mu}(t,\tau)\,q_{s_\mu}(\tau)\,d\tau = R_{p_i}(t) \qquad (t\geqslant t_1) \qquad (i=1,\ldots,m) \tag{1.23}$$

The system of integral equations (1.23) can be solved by means of a numerical method. To this end let us divide the time interval (t_1, t) into small intervals (t_1, t_2), (t_2, t_3), (t_3, t_4), ... and look for $q_{s_\mu}(t)$ $(\mu = 1, \ldots, m)$ in the form of step functions, having constant values in each of the above intervals.

In doing so we shall obtain (from 1.23) the following system of recurrence equations:

$$\sum_{\mu=1}^{m}\left[\int_{t_0}^{t_1} W_{p_i s_\mu}(t_1, \tau)\, d\tau\right] q_{s_\mu}(t_0) = R_{p_i}(t_1) \qquad (i = 1, \ldots, m)$$

$$\sum_{\mu=1}^{m}\left[\int_{t_0}^{t_1} W_{p_i s_\mu}(t_2, \tau)\, d\tau\right] q_{s_\mu}(t_0) + \qquad\qquad (1.24)$$

$$+ \sum_{\mu=1}^{m}\left[\int_{t_1}^{t_2} W_{p_i s_\mu}(t_2, \tau)\, d\tau\right] q_{s_\mu}(t_1) = R_{p_i}(t_2) \qquad (i = 1, \ldots, m)$$

$$\sum_{\mu=1}^{m}\left[\int_{t_0}^{t_1} W_{p_i s_\mu}(t_3, \tau)\, d\tau\right] q_{s_\mu}(t_0) + \sum_{\mu=1}^{m}\left[\int_{t_1}^{t_2} W_{p_i s_\mu}(t_3, \tau)\, d\tau\right] q_{s_\mu}(t_1) +$$

$$+ \sum_{\mu=1}^{m}\left[\int_{t_2}^{t_3} W_{p_i s_\mu}(t_3, \tau)\, d\tau\right] q_{s_\mu}(t_2) = R_{p_i}(t_3) \qquad (i = 1, \ldots, m)$$

. . . .

From (1.24) we can find in succession the following quantities:

$$q_{s_\mu}(t_0), \quad q_{s_\mu}(t_1), \quad q_{s_\mu}(t_2), \quad \ldots \qquad (\mu = 1, \ldots, m)$$

which are the values of the desired step functions $q_{s_\mu}(t)$ $(\mu = 1, \ldots, m)$ during time intervals (t_0, t_1), (t_1, t_2), (t_2, t_3), ..., respectively.

Let us now consider a case when the number of additional external forces is less than the number of the generalized coordinates which must vary with respect to time in accordance with the prescribed law.

The relationships (1.23) for this case will be satisfied only at discrete instants of time $t = T_1, T_2, T_3, \ldots$. For stationary systems this was shown by Lerner [6].

Thus, if there is only one additional force $q_s(t)$ and the conditions (1.19) still must be satisfied, the relationships (1.23) at instant $t = T_1$ will assume the form:

$$\int_{t_0}^{T_1} W_{p_i s}(T_1, \tau)\, q_s(\tau)\, d\tau = R_{p_i}(T_1) \qquad (i = 1, \ldots, m) \qquad (1.25)$$

Let us divide the interval (t_0, T_1) into m equal or unequal subintervals (t_0, t_1), (t_1, t_2), (t_2, t_3), ..., (t_{m-1}, T). Let us assume that $q_s(t)$ is a step function and denote its values in these intervals by $q_s(t_0)$, $q_s(t_1)$, ..., $q_s(t_{m-1})$, respectively.

Let us introduce the following notations:

$$\int_{t_0}^{t_1} W_{p_i s}(T_1, \tau)\, d\tau = c_i^{(0)}, \quad \int_{t_1}^{t_2} W_{p_i s}(T_1, \tau)\, d\tau = c_i^{(1)}, \ldots, \int_{t_{m-1}}^{T_1} W_{p_i s}(T_1, \tau)\, d\tau = c_i^{(m-1)}$$

$$(i = 1, \ldots, m) \qquad (1.26)$$

Now relationships (1.25) assume the form:

$$c_i^{(0)} q_s(t_0) + c_i^{(1)} q_s(t_1) + \ldots + c_i^{(m-1)} q_s(t_{m-1}) = R_{p_i}(T_1) \qquad (i = 1, \ldots, m) \quad (1.27)$$

From the system of linear nonhomogenious algebraic equations (1.27) we find the values of $q_s(t_0)$, $q_s(t_1)$, ..., $q_s(t_{m-1})$, i.e. we find the law of variation of additional external forces which assures that at instant $t = T_1$ the following conditions will be satisfied

$$z_{p_i}(T_1) = r_{p_i}(T_1) \qquad (i = 1, \ldots, m) \qquad (1.28)$$

At $t = T_2$ the relationships (1.23) will assume the form:

$$\int_{t_0}^{T_2} W_{p_i s}(T_2, \tau)\, q_s(\tau)\, d\tau = R_{p_i}(T_2) \qquad (i = 1, \ldots, m) \qquad (1.29)$$

or

$$\int_{T_1}^{T_2} W_{p_i s}(T_2, \tau)\, q_s(\tau)\, d\tau = R_{p_i}^{*}(T_2) \qquad (i = 1, \ldots, m) \qquad (1.30)$$

where

$$R_{p_i}^{*}(T_2) = R_{p_i}(T_2) - \int_{t_0}^{T_1} W_{p_i s}(T_2, \tau)\, q_s(\tau)\, d\tau \qquad (i = 1, \ldots, m) \qquad (1.31)$$

The interval T_1, T_2 will be also divided into m equal of unequal subintervals (t_m, t_{m+1}), (t_{m+1}, t_{m+2}), ..., (t_{2m-1}, T_2), where

$$t_m = T_1 \qquad (1.32)$$

The values of the step function $q_s(t)$ in the intervals (t_m, t_{m+1}), (t_{m+1}, t_{m+2}), ..., (t_{2m-1}, T_2) will be denoted by $q_s(t_m)$, $q_s(t_{m+1})$, ..., $q_s(t_{2m-1})$, respectively. Let us use the analogous notation

$$\int_{t_m}^{t_{m+1}} W_{p_is}(T_2,\tau)\,d\tau = c_i^{(m)}, \qquad \int_{t_{m+1}}^{t_{m+2}} W_{p_is}(T_2,\tau)\,d\tau = c_i^{(m+1)}, \ldots$$

$$\ldots, \int_{t_{2m-1}}^{T_2} W_{p_is}(T_2,\tau)\,d\tau = c_i^{(2m-1)} \qquad (i = 1,\ldots,m) \tag{1.33}$$

The relationships (1.30) will become: (1.34)

$$c_i^{(m)}q_s(t_m) + c_i^{(m+1)}q_s(t_{m+1}) + \ldots + c_i^{(2m-1)}q_s(t_{2m-1}) = R_{p_i}{}^*(T_2) \quad (i = 1,\ldots,m)$$

From the system of equations (1.34) we find the values of $q_s(t_m)$, $q_s(t_{m+1})$, ..., $q_s(t_{2m-1})$, i.e. we find the law of variation of the additional external force in the interval (T_1, T_2), which assures that at instant T_2 the following conditions are satisfied

$$z_{p_i}(T_2) = r_{p_i}(T_2) \qquad (i = 1,\ldots,m) \tag{1.35}$$

This process may be continued, and thus to assure that in the presence of only one additional force $q_s(t)$ the m conditions of (1.19) will be satisfied at instants $t^* = T_1, T_2, T_3, \ldots$:

$$z_{p_i}(t^*) = r_{p_i}(t^*) \qquad (i = 1,\ldots,m)$$

Let us note that the values of $q_s(t_j)$, generally speaking, will increase as the length of the time intervals $T_i - T_{i-1}$ is decreased.

For fixed values of T_i - at which conditions (1.19) must be satisfied - a problem may be given in terms of some extremum properties of motion such as minimum root-mean-square deviation $r_{pi}(t) - z_{pi}(t)$, or others. These problems will reduce to the problem of conditional extremum of corresponding functionals, which may be determined from (1.23).

Let us make also the following observation. In equations (1.24), (1.25), and (1.29), the functions $W_{jl}(t, \tau)$, for fixed value of time $t = t_\zeta$, are assumed to be known. As is seen from (1.17), in order to determine the functions $W_{jl}(t_\zeta, \tau)$ one must know the functions $N_{j\xi}(t_\zeta, \tau)$, which are elements of the matrix weighting function $N(t, \tau)$ at the same instant $t = t_\zeta$. As is known,

$$N_{j\xi}(t_\zeta,\tau) = Z_\xi(\tau) \tag{1.36}$$

where $Z_\xi(\tau)$ are solutions of the conjugate system of equations

$$\frac{dZ_\xi}{d\tau} - \sum_{k=1}^{r} a_{k\xi}(\tau) Z_k = 0 \qquad (\xi = 1,\ldots,r) \tag{1.37}$$

constructed for the system of equations (1.10) and which at $\tau = t_\zeta$ *assume* the following values:

$$Z_j(t_\zeta) = 1, \qquad Z_k(t_\zeta) = 0 \qquad (k = 1, 2, \ldots, j-1,\ j+1, \ldots, r) \tag{1.38}$$

Thus, to make possible the determination of the functions $W_{ij}(t_\zeta, r)$, it is necessary to solve the conjugate system of equations (1.37), subject to the conditions (1.38). From this follows the method of determination of additional external forces $q_s(t)$ by means of electronic computers.

2. Impulsive systems. For this case, equations of motion of a nonstationary impulsive system will be represented by a system of linear difference equations

$$\sum_{k=1}^{n} f_{jk}(T)\, y_k = x_j(t) + q_j(t) \qquad (j = 1, \ldots, n) \tag{2.1}$$

where T is the lead operator defined by the following relationship

$$T^s y_k = y_k(t + s\tau) \tag{2.2}$$

and τ is some constant.

Equations 2.1) may be obtained from (1.1) by replacing the differential operator D by the lead operator T. Having carried out the transformations in (1.2) to (1.9), we shall reduce the system of equations (2.1) to the form

$$Tz_j + \sum_{k=1}^{r} a_{jk}(t)\, z_k = X_j(t) + Q_j(t) \qquad (j = 1, \ldots, r) \tag{2.3}$$

where, analogously to (1.5),

$$z_1 = y_1, \quad z_2 = Ty_1, \ldots, \quad z_{m_1} = T^{m_1 - 1} y_1, \ldots, \quad z_r = T^{m_n - 1} y_n \tag{2.4}$$

and the functions $X_j(t)$ and $Q_j(t)$ are defined by the expressions (1.7) and (1.8).

The system of scalar equations (2.3) is equivalent to the matrix equation

$$Tz + a(t) z = X(t) + Q(t) \tag{2.5}$$

where the matrices z, $a(t)$, $X(t)$, and $Q(t)$ are defined in (1.12). Solution of the equation (2.5) is of the form:

$$z(t) = \theta(t)\,\theta^{-1}(t - \vartheta\tau)\, z^*(t - \vartheta\tau) +$$

$$+ \sum_{j=1}^{\vartheta} \theta(t)\,\theta^{-1}(t - \vartheta\tau + j\tau)\,[X(t - \vartheta\tau + j\tau - \tau) + Q(t - \vartheta\tau + j\tau - \tau)] \tag{2.6}$$

where $\theta(t)$ is a square matrix, the columns of which are linearly independent solutions of the following homogeneous matrix equation

$$Tz + a(t)z = 0 \tag{2.7}$$

The matrix $\theta^{-1}(t)$ is the inverse matrix of $\theta(t)$ and denotes the integral part of t/r.

In the expression (2.6) the second term becomes zero in the interval $0 < t < r$ so that, according to (2.6)

$$z(t) = z^*(t) \qquad (0 < t < \tau) \tag{2.8}$$

where $z^*(t)$ is a given matrix defined in the interval $0 < t < r$ by the law of variation of the sought matrix $z(t)$ in this (initial) time interval. In accordance with (2.6), the elements of the matrix $z(t)$ will be of the form:

$$z_\nu(t) = \sum_{k=1}^{r} [\theta(t)\,\theta^{-1}(t-\vartheta\tau)]_{\nu k}\, z_k^*(t-\vartheta\tau) + \tag{2.9}$$

$$+ \sum_{k=1}^{r} \sum_{j=1}^{\vartheta} [\theta(t)\,\theta^{-1}(t-\vartheta\tau+j\tau)]_{\nu k}\, [X_k(t-\vartheta\tau+j\tau-\tau) +$$

$$+ Q_k(t-\vartheta\tau+j\tau-\tau)] \qquad (\nu = 1, \ldots, r)$$

Denoting the matrix weighting function by $N(t, jr)$

$$N(t, j\tau) = \theta(t)\,\theta^{-1}(t-\vartheta\tau+j\tau) \tag{2.10}$$

the solution (2.6) can be written as follows

$$z(t) = N(t, 0)\, z^*(t-\vartheta\tau) +$$

$$+ \sum_{j=1}^{\vartheta} N(t, j\tau)\, [X(t-\vartheta\tau+j\tau-\tau) + Q(t-\vartheta\tau+j\tau-\tau)] \tag{2.11}$$

The expressions (2.9) will now assume the form:

$$z_\nu(t) = \sum_{k=1}^{r} N_{\nu k}(t, 0)\, z_k^*(t-\vartheta\tau) + \tag{2.12}$$

$$+ \sum_{k=1}^{r} \sum_{j=1}^{\vartheta} N_{\nu k}(t, j\tau)\, [X_k(t-\vartheta\tau+j\tau-\tau) + Q_k(t-\vartheta\tau+j\tau-\tau)] \quad (\nu = 1, \ldots, r)$$

Since the functions $X_\mu(t)$ and $Q_\mu(t)$ with $\mu \neq \sigma_l (l = 1, \ldots, n)$ are identically equal to zero, the expressions (2.12) may be rewritten as follows:

$$z_\nu(t) = \sum_{k=1}^{r} N_{\nu k}(t, 0)\, z_k^*(t-\vartheta\tau) +$$

$$+ \sum_{i=1}^{n} \sum_{j=1}^{\vartheta} N_{\nu\sigma_i}(t, j\tau)\, [X_{\sigma_i}(t-\vartheta\tau+j\tau-\tau) + Q_{\sigma_i}(t-\vartheta\tau+j\tau-\tau)] \tag{2.13}$$

$$(\nu = 1, \ldots, r)$$

Replacing X_{σ_i} and Q_{σ_i} in accordance with (1.7), let us reduce (2.13) to the following:

$$z_\nu(t) = \sum_{k=1}^{r} N_{\nu k}(t, 0)\, z_k^*(t-\vartheta\tau) + \tag{2.14}$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{\vartheta}\sum_{l=1}^{n} N_{\nu\sigma_i}(t, j\tau)\,\frac{B_{li}(t-\vartheta\tau+j\tau-\tau)}{\Delta^*(t-\vartheta\tau+j\tau-\tau)}\,[x_l(t-\vartheta\tau+j\tau-\tau)+$$

$$+q_l(t-\vartheta\tau+j\tau-\tau)] \qquad (\nu=1,\ldots,r)$$

Writing

$$W_{\nu l}(t, j\tau) = \sum_{i=1}^{n} N_{\nu\sigma_i}(t, j\tau)\,\frac{B_{li}(t-\vartheta\tau+j\tau-\tau)}{\Delta^*(t-\vartheta\tau+j\tau-\tau)} \qquad \begin{pmatrix}\nu=1,\ldots,r\\ l=1,\ldots,n\end{pmatrix} \tag{2.15}$$

the expressions (2.14) may be represented in the form:

$$z_\nu(t) = \sum_{k=1}^{r} N_{\nu k}(t, 0)\, z_k^*(t-\vartheta\tau) + \tag{2.16}$$

$$+\sum_{l=1}^{n}\sum_{j=1}^{\vartheta} W_{\nu l}(t, j\tau)\,[x_l(t-\vartheta\tau+j\tau-\tau)+q_l(t-\vartheta\tau+j\tau-\tau)] \qquad (\nu=1,\ldots,r)$$

To satisfy the conditions (1.19) we shall have the following relationships, analogous to (1.23):

$$\sum_{\mu=1}^{m}\sum_{j=1}^{\vartheta} W_{p_i s_\mu}(t, j\tau)\, q_{s_\mu}(t-\vartheta\tau+j\tau-\tau) = R_{p_i}(t) \quad (t \geqslant t_1) \quad (i=1,\ldots,m), \tag{2.17}$$

where

$$R_{p_i}(t) = r_{p_i}(t) - \sum_{k=1}^{r} N_{p_i k}(t, 0)\, z_k^*(t-\vartheta\tau) - \tag{2.18}$$

$$-\sum_{l=1}^{n}\sum_{j=1}^{\vartheta} W_{p_i l}(t, j\tau)\, x_l(t-\vartheta\tau+j\tau-\tau) \qquad (j=1,\ldots,m)$$

Relationships (2.17) define the law, in accordance with which additional external forces $q_{s_\mu}(t)$ $(\mu = 1, \ldots, m)$ must vary in order for some generalized coordinates of the system z_{p_i} $(i = 1, \ldots, m)$ after some instant of time $t_1 = j_1\tau$ (t_1 is some multiple of τ) to vary in accordance with the prescribed laws (1.19):

$$z_{p_i}(t) = r_{p_i}(t) \qquad (t \geqslant t_1) \qquad (i=1,\ldots,m)$$

To solve the system of equations (2.17) one can, as in the preceding section, use numerical methods. To this end let us divide the time interval $(j_1\tau,\ \tau)$ into small intervals $(j_1\tau,\ j_2\tau)$, $(j_2\tau,\ j_3\tau)$ $(j_3\tau,\ j_4\tau)$, ... and look for $q_{s_\mu}(t)$ in the form of step functions having constant values in these intervals. Let us denote these values by $q_{s_\mu}(0)$, $q_{s_\mu}(t_1)$, $q_{s_\mu}(t_2)$, $q_{s_\mu}(t_3)$, respectively. Here

$$t_1 = j_1\tau, \qquad t_2 = j_2\tau, \qquad t_3 = j_3\tau, \ldots \tag{2.19}$$

In doing so we obtain from (2.17) the following system of recurrence relationships:

$$\sum_{\mu=1}^{m}\Big[\sum_{j=1}^{j_1} W_{p_i s_\mu}(t_1, j\tau)\Big] q_{s_\mu}(0) = R_{p_i}(t_1) \qquad (i = 1, \ldots, m) \tag{2.20}$$

$$\sum_{\mu=1}^{m}\Big[\sum_{j=1}^{j_1} W_{p_i s_\mu}(t_2, j\tau)\Big] q_{s_\mu}(0) + \sum_{\mu=1}^{m}\Big[\sum_{j=j_1+1}^{j_2} W_{p_i s_\mu}(t_2, j\tau)\Big] q_{s_\mu}(t_1) = R_{p_i}(t_2) \qquad (i = 1, \ldots, m)$$

$$\sum_{\mu=1}^{m}\Big[\sum_{j=1}^{j_1} W_{p_i s_\mu}(t_3, j\tau)\Big] q_{s_\mu}(0) + \sum_{\mu=1}^{m}\Big[\sum_{j=j_1+1}^{j_2} W_{p_i s_\mu}(t_3, j\tau)\Big] q_{s_\mu}(t_1) +$$

$$+ \sum_{\mu=1}^{m}\Big[\sum_{j=j_2+1}^{j_3} W_{p_i s_\mu}(t_3, j\tau)\Big] q_{s_\mu}(t_2) = R_{p_i}(t_3) \qquad (i = 1, \ldots, m)$$

. . . .

From (2.20) we determine the numbers

$$q_{s_\mu}(0), \; q_{s_\mu}(t_1), q_{s_\mu}(t_2), \ldots \qquad (\mu = 1, \ldots, m)$$

Then the number of additional external forces is less than the number of coordinates, the variation of which with respect to time must be in accordance with the prescribed law, the relationships (2.17), as in the continuous system, can be satisfied only at discrete instants $t = T_1$, T_2, T_3, ..., which for the sake of simplicity, will be taken as some multiples of τ:

$$T_1 = \vartheta_1\tau, \qquad T_2 = \vartheta_2\tau, \qquad T_3 = \vartheta_3\tau, \ldots \tag{2.21}$$

In the presence of only one additional external force $q_s(t)$, the relationships (2.17) at $t = T_1$ assume the form:

$$\sum_{j=1}^{\vartheta_1} W_{p_i s}(T_1, j\tau)\, q_s(j\tau - \tau) = R_{p_i}(T_1) \qquad (i = 1, \ldots, m) \tag{2.22}$$

Let us divide the interval $(0, T_1)$ into m equal or unequal sub-intervals $(0, j_1\tau)$, $(j_1\tau, j_2\tau)$, $(j_2\tau, j_3\tau)$, ..., $(j_{m-1}\tau, T_1)$ and, assuming that $q_s(t)$ is a step function, let us denote its values in these intervals by $q_s(0)$, $q_s(t_1)$, ..., $q_s(t_{m-1})$, respectively. Furthermore, let us use the notation

$$\sum_{j=1}^{j_1} W_{p_is}(T_1,\ j\tau) = c_i^{(0)}, \qquad \sum_{j=j_1+1}^{j_2} W_{p_is}(T_1,\ j\tau) = c_i^{(1)}, \ldots$$
$$\ldots, \sum_{j=j_{m-1}+1}^{\vartheta_1} W_{p_is}(T_1,\ j\tau) = c_i^{(m-1)} \quad (i = 1, \ldots, m) \tag{2.23}$$

The relationships (2.22) assume the form:

$$c_i^{(0)} q_s(0) + c_i^{(1)} q_s(t_1) + \ldots + c_i^{(m-1)} q_s(t_{m-1}) = R_{p_i}(T_1) \quad (i = 1, \ldots, m) \tag{2.24}$$

The values of $q_s(0)$, $q_s(t_1)$, ..., $q_s(t_{m-1})$, defining the law of variation of additional external force $q_s(t)$ in the interval $(0,\ T_1)$, will be found from (2.24). Continuing this process, as in (1.29)-(1.34), we can determine the law of variation of the additional external force $q_s(t)$ in intervals $(T_1,\ T_2)$, $(T_2,\ T_3)$, etc. Thus, the fulfilment of the conditions (1.19)

$$z_{p_i}(t^*) = r_{p_i}(t^*) \qquad (i = 1, \ldots, m)$$

is assured at discrete instant $t = T_1,\ T_2,\ T_3,\ \ldots$

In relationships (2.20)-(2.23) the functions $W_{\nu l}(t,\ j\tau)$, for fixed value of time $t = t_\zeta$, are assumed to be known. To determine the functions $W_{\nu l}(t,\ j\tau)$ in accordance with (2.15), one must know the functions $N_{\nu\xi}(t_\zeta,\ j\tau)$ which are elements of the matrix weighting function (2.10) at $t = t_\zeta$. As is shown in [10]:

$$N_{\nu\xi}(t_\zeta,\ j\tau) = Z_\xi(t_\zeta - \vartheta_\zeta\tau + j\tau) \tag{2.25}$$

where ζ is the integral part of t_ζ/τ and Z_ξ are solutions of a conjugate system of difference equations

$$Z_\xi(t) + \sum_{k=1}^{k} a_{k\xi} Z_k(t+\tau) = 0 \qquad (\xi = 1, \ldots, r) \tag{2.26}$$

constructed for the system of difference equations (2.3) and satisfying the following conditions in the interval $\zeta\tau < t < (\zeta + 1)/\tau$

$$Z_\nu^\tau(t) = 1, \qquad Z_k(t) = 0 \quad (k = 1,\ 2, \ldots, \nu-1,\ \nu+1, \ldots, r) \tag{2.27}$$

Thus, to make possible the determination of the functions $W_{\nu l}(t_\zeta,\ j\tau)$, it is necessary to find the solution of the system of homogeneous difference equations (2.26), subject to conditions (2.27).

From this follows also the method of solution of the problem considered here for impulsive systems by means of electronic computers.

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